Uniqueness of Generalized Quadrature Domains

via the Faber Transform

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(joint work with Nikolai Makarov)

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Quadrature Identities

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Epstein & Schiffer (1965): $\mathbb{D}_r(w_0)$ is the only[2] domain with this property.

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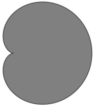

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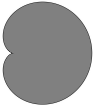

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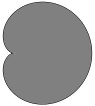

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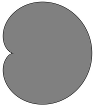

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These are examples of *quadrature identities*.

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Definition 1.1 (Bounded Quadrature domain)

A bounded domain $\Omega \subset \widehat{\mathbb{C}}$ is a *bounded* QD if there exists $h \in \text{Rat}_0(\Omega)$ s.t.[3][4]

$$\int_\Omega f dA = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw$$

$\forall f \in \mathcal{A}(\Omega)$. This is denoted by $\Omega \in \text{QD}(h)$. (we also assume $\infty \notin \partial\Omega$)

[3] $\text{Rat}(\Omega)$ = space of rational functions analytic in Ω^c. (all poles are in Ω)
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Can also write: $\displaystyle\int_\Omega f dA = \mu(f)$, where $\displaystyle\mu = \bar{\partial} h = \sum_{k,j} c_{k,j}(-1)^{n_j+1}\delta_{p_k}^{(n_j)}$

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where $f(w) = \sum_{j=1}^{\infty} f_j w^{-j}$.

Remark: $\Omega \subset \widehat{\mathbb{C}}$ is a QD iff it admits a *Schwarz function* $S : \Omega \to \widehat{\mathbb{C}}$.

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$$C^{\Omega^c}(w) = \lim_{r \to \infty} \frac{1}{\pi} \int_{\Omega^c \cap \mathbb{D}_r} \frac{dA(\xi)}{w - \xi}$$

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A $S-$function is a continuous map

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such that $S \in \mathcal{M}(\Omega)$ and[5]

$$S(w) \doteq \overline{w}$$

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Also,

$$S(w) = h(w) + C^{\Omega^c}(w), \quad w \in \Omega$$

(where C^{Ω^c} is understood in terms of its Cauchy principal value)[6]

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Unbounded QD Example: The Deltoid

The complement of the deltoid is an unbounded quadrature domain, $\Omega = \left\{z + \frac{1}{2z^2} : |z| > 1\right\} \in \mathrm{QD}\left(\frac{w^2}{2}\right)$:

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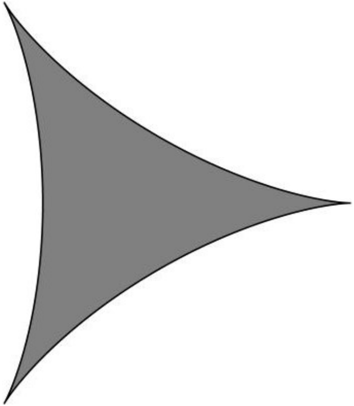

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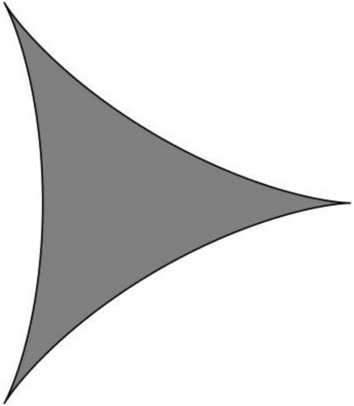

$$\int_\Omega f dA = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w) \frac{w^2}{2} dw = \frac{1}{2} f_3$$

The Uniqueness Problem for QDs

Question: When is a QD uniquely associated to its quadrature function?

i.e. For what rational functions h is there a unique s.c. domain $\Omega \in \text{QD}(h)$?

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Logarithmic Potential Theory

Let μ be a compactly supported non-negative measure on $\mathbb{C}$ and $Q : \mathbb{C} \to (-\infty, \infty]$ an *admissible*[10] external potential. The logarithmic potential and energy associated to μ, Q are respectively

$$U_Q^\mu(w) = \int_{\mathbb{C}} \left(\ln \frac{1}{|w-\xi|} + Q(\xi) + Q(w) \right) d\mu(\xi), \qquad I_Q(\mu) = \int_{\mathbb{C}} U_Q^\mu d\mu.$$

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Frostman: Under these conditions, for each $t > 0$, there exists a unique compactly supported measure μ_t for which

$$I_Q(\mu_t) = \gamma := \inf_{\mu:\mu(1)=t} I_Q(\mu).$$

Moreover, $d\mu_t = \frac{\Delta Q}{2\pi} \mathbb{1}_{K_t}$, where $K_t = \text{supp}(\mu_t)$. μ_t is called the *equilibrium measure* of mass t associated to Q (also called a droplet).

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Note: If Q doesn't satisfy the growth condition, one can "localize" by taking $Q_X(w) = Q(w)\mathbb{1}_X + \infty \mathbb{1}_{X^c}$ for an appropriate compact $X \subset \mathbb{C}$. If Q_X is real analytic in a nbhd of K_t, we call K_t a *local droplet*.

---

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Lemma 2.1

Let K_t be a local droplet corresponding to a "Hele-Shaw" potential, $Q(w) = |w|^2 - 2Re(H(w))$, where $h = H'$ is a rational function. Then K_t^c is a disjoint union of QDs, with $h =$ the sum of their quadrature functions.

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Duality of Quadrature Domains and Local Droplets

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Conversely, if the complement of some compact set K_t is a disjoint union of QDs with $h =$ the sum of their quadrature functions, then K_t is a local droplet of a potential of the form

$$Q(w) = |w|^2 - 2Re\left(\sum_l \mathbb{1}_{K_l} H_l(w)\right), \quad H_l(w) = c_l + \int_{w_l}^{w} h(\xi)d\xi,$$

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$$\text{QD} \xleftrightarrow{\text{lemmas}} \text{local droplet} \xleftrightarrow{\text{Frostman}} \text{uniqueness}$$

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Example

Recall that the complement of the deltoid,

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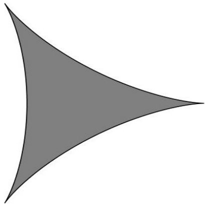

The deltoid

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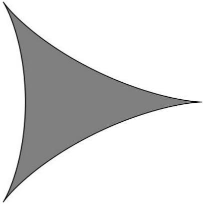
Then by the lemma, the deltoid $K := \Omega^c$ is a local droplet of the potential

$$Q(w) = |w|^2 - 2\mathrm{Re}\left(\int_0^w \frac{\xi^2}{2} d\xi\right)$$
$$= |w|^2 - 2\mathrm{Re}\left(\frac{w^3}{6}\right)$$



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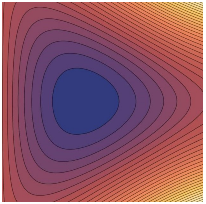

Contour plot for Q

The Faber Transform

Let $\Omega \subset \widehat{\mathbb{C}}$ be unbounded and simply connected with Riemann map $\varphi : \mathbb{D}^{-} \to \Omega$,

$$\varphi(z) = az + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad a = \text{rad}_\infty(\Omega) > 0$$

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The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega^-)$,[12]

$$\Phi_\varphi(f)(w) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}^-} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f \circ \psi(\xi)}{\xi - w} d\xi$$

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Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

$$\Phi_\varphi(z^n)(w) =: F_n(w) \ \ (n\text{th Faber polynomial})$$

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$$\varphi(z) = az + f_0 + f_1 z^{-1} + f_2 z^{-2} + \cdots , \qquad a = \mathrm{rad}_\infty(\Omega) > 0$$

The associated *exterior Faber transform* Φ_φ is a linear iso $\mathcal{A}(\mathbb{D}) \to \mathcal{A}(\Omega^-)$,[12]

$$\Phi_\varphi(f)(w) = \frac{1}{2\pi i} \oint_{\partial\mathbb{D}^-} \frac{f(z)\varphi'(z)}{\varphi(z) - w} dz = \frac{1}{2\pi i} \oint_{\partial\Omega} \frac{f \circ \psi(\xi)}{\xi - w} d\xi$$

Note: the Faber transform preserves polynomials and rational functions, e.g.

$$\Phi_\varphi\left(\frac{1}{z - z_0}\right)(w) = \frac{\varphi'(z_0)}{w - \varphi(z_0)},$$

$$\Phi_\varphi\left(z^n\right)(w) =: F_n(w) \quad (n\text{th Faber polynomial})$$

The *interior* Faber transform, defined for bounded s.c. domains is defined similarly, and is a map $\Phi_\varphi : \mathcal{A}_0(\mathbb{D}^-) \to \mathcal{A}_0(\Omega^-)$.

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The Faber Transform Method

If $\Omega \in \mathrm{QD}(h)$ is bounded s.c, with Riemann map φ, then φ is rational and[13]

$$h = \Phi_\varphi\left(\varphi^\#\right) \qquad \text{Chang \& Makarov (2013)}$$

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We've shown that if $\Omega \in QD(h)$ is unbounded and s.c, then the associated Riemann map φ is rational and satisfies

$$h = \Phi_{\varphi}\left(\varphi^{\#}\right),$$

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Main Result: Existence and Uniqueness of One Pt. UQDs

While the uniqueness of one point bounded QDs is well understood, the situation for general one point *unbounded* QDs has received little attention in the literature.

[15] The $c = 0$ and $w_0 = 0$ cases are trivial.

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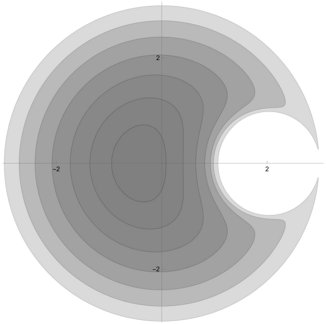

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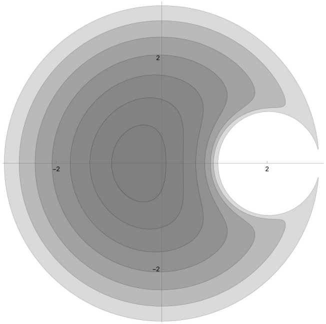

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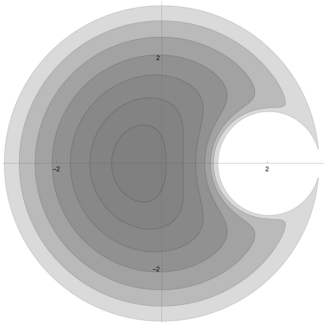
By change of variables,

$$\Omega \in QD\left(\frac{c}{w - w_0}\right) \iff \frac{2}{w_0}\Omega \in QD\left(\frac{\frac{4c}{|w_0|^2}}{w - 2}\right)$$

so we can wlog assume $w_0 = 2$.



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Theorem 4.1

If $c \in \mathbb{C} \setminus \mathbb{R}$ and $w_0 \neq 0$, then there exists a simply connected domain $\Omega \in QD\left(\frac{c}{w-w_0}\right)$ if and only if $|w_0|^2 + 2(Re(c) - |c|) > 0$. In this case, Ω is the unique such domain of its conformal radius.

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Set $A(\Omega_t) = t$. Then for each such w_0 and c,

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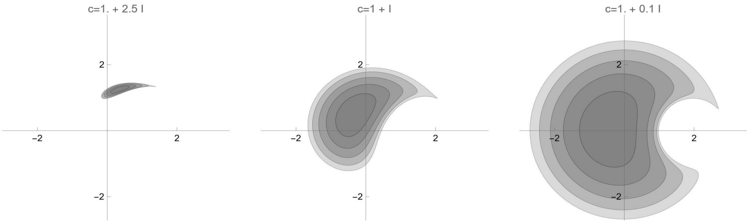

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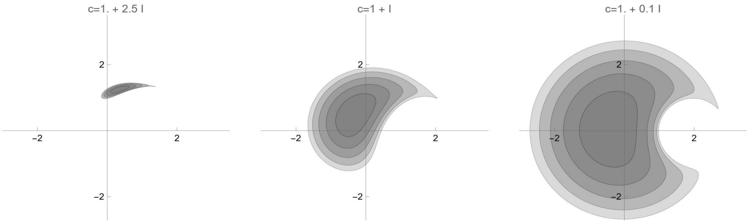

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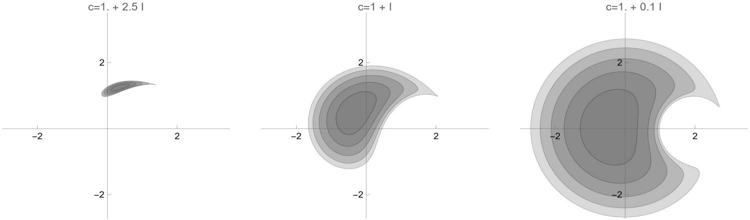

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Outline of Proof

$$\Omega_t \in \mathrm{QD}(h), \qquad h(w) = \frac{c}{w-2}$$

Key tool: Correspondence between UQDs and local droplets.
Can consider the associated potential:

$$Q(w) = |w|^2 - 2\mathrm{Re}\,(H(w)) \quad \left(H(w) = c\ln(w-2), \quad H'(w) = h(w) = \frac{c}{w-2}\right).$$

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3. Cusp development: Use Faber transform to obtain representation of Riemann map $\varphi_t : \mathbb{D}^- \to \Omega_t$, then use representation of Riemann map to demonstrate that $\varphi_t'(z) = 0$ for some $z \in \partial\mathbb{D}$ and $t > 0$.

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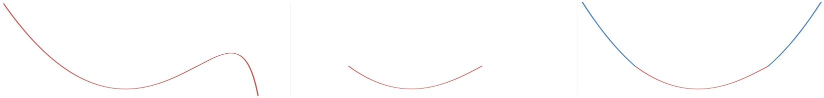

singular potential *localized potential* *modified potential*

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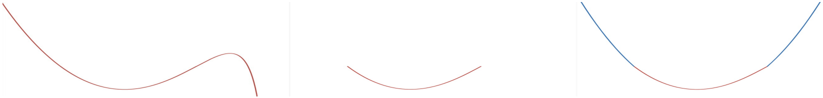

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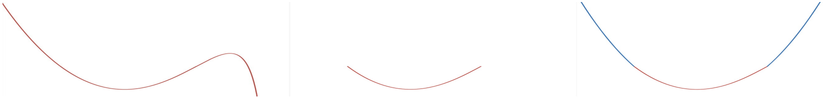

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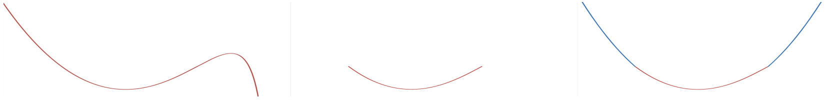

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Calculus exercise: Show that $Q(w) = |w|^2 - 2\text{Re}(c \ln(w-2))$ has a local minimum $z_0 \neq 2$ iff $2 + \text{Re}(c) - |c| > 0$ and the minimum is unique when it exists.

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4. Finally, by localizing to an open nbhd of $K_t \cup K'_t$ and applying Frostman's theorem, we find that $K_t = K'_t$, so $\Omega_t = \Omega'_t$.

Cusp Formation

Recall that if $\Omega \in \mathrm{QD}(h)$ is s.c. and unbounded then there exists $a > 0$ such that $\varphi(z) = az + \Phi_{\varphi}^{-1}(h)^{\#}(z)$.

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Applying the additional Faber transform relation, $h = \Phi_\varphi\left(\varphi^\#\right)$, we obtain

$$\frac{c}{w-2} = h(w) = \Phi_\varphi\left(\varphi^\#\right)(w) = \frac{1}{w-2}\frac{(az_0|z_0|^2 - 2)(a\overline{z_0} - 2)}{|z_0|^2}.$$

Proof of Cusp Formation

$$c = (az_0|z_0|^2 - 2)(a\overline{z_0} - 2)|z_0|^{-2}, \qquad \overline{c} = (a\overline{z_0}|z_0|^2 - 2)(az_0 - 2)|z_0|^{-2}$$

Considering the obtained equation along with its complex conjugate and eliminating $\overline{z_0}$, we obtain a sextic in z_0,

$$0 = z_0^6 + z_0^5\left(\frac{a}{2} + O(1)\right) + z_0^4 O(1) + z_0^3 O\left(a^{-1}\right) + z_0^2 O(1) + z_0 O\left(a^{-1}\right) + 4a^{-2}.$$

[16] Y. Ameur, N-G. Kang, N. Makarov., "Scaling limits of random normal matrix processes at singular boundary points". (2020)

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The former is impossible because $|z_0| > 1$, and the latter is because then

$$c = \frac{(az_0|z_0|^2 - 2)(a\overline{z_0} - 2)}{|z_0|^2} = \frac{\left(\frac{a^4}{8} + O(a^3)\right)\left(\frac{a^2}{2} + O(a)\right)}{\frac{a^2}{4} + O(a)} = \frac{a^4}{4} + O(a^3)$$

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Proof of Cusp Formation

$$c = (az_0|z_0|^2 - 2)(a\overline{z_0} - 2)|z_0|^{-2}, \qquad \overline{c} = (a\overline{z_0}|z_0|^2 - 2)(az_0 - 2)|z_0|^{-2}$$

Considering the obtained equation along with its complex conjugate and eliminating $\overline{z_0}$, we obtain a sextic in z_0,

$$0 = z_0^6 + z_0^5\left(\frac{a}{2} + O(1)\right) + z_0^4 O(1) + z_0^3 O\left(a^{-1}\right) + z_0^2 O(1) + z_0 O\left(a^{-1}\right) + 4a^{-2}.$$

Asymptotic analysis for $a \to \infty$ tells us that either $z_0 \xrightarrow{a\to\infty} 0$ or $z_0 = \frac{a}{2} + O(1)$.

The former is impossible because $|z_0| > 1$, and the latter is because then

$$c = \frac{(az_0|z_0|^2 - 2)(a\overline{z_0} - 2)}{|z_0|^2} = \frac{\left(\frac{a^4}{8} + O(a^3)\right)\left(\frac{a^2}{2} + O(a)\right)}{\frac{a^2}{4} + O(a)} = \frac{a^4}{4} + O(a^3)$$

Thus $\nexists\Omega_t$ for t sufficiently large, so there exists a maximal t. By Sakai's theorem[16], $\partial\Omega_{t_*}$ must contain a $(\nu, 2)-$cusp, where $\nu \in 3 + 4\mathbb{N}_0$. That $\nu = 3$ follows from analysis of φ_{t_*}.

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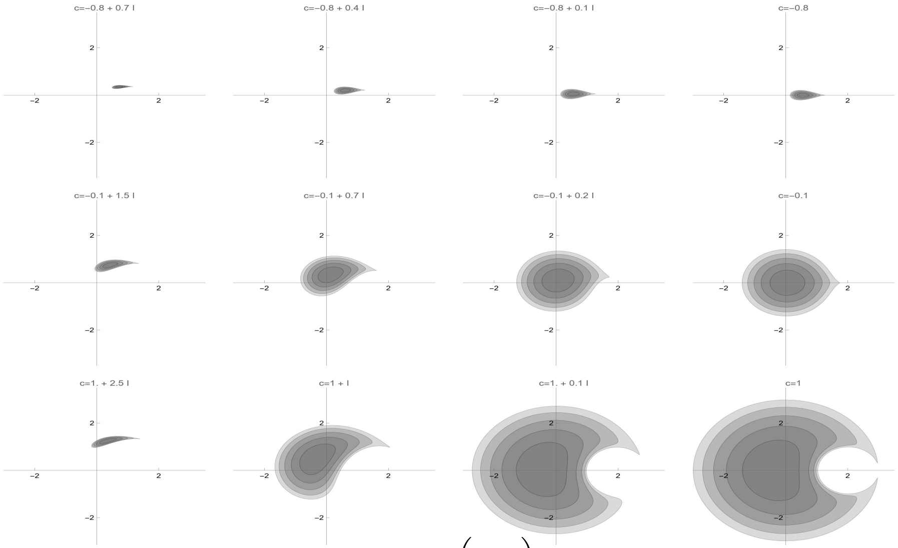
Some Examples



$$\Omega \in \mathrm{QD}\left(\frac{c}{w-2}\right)$$

Power-Weighted Quadrature Domains

Definition 5.1 (Power-Weighted Quadrature Domain)

We call a bounded (resp. unbounded) domain $\Omega \subset \widehat{\mathbb{C}}$ a *power-weighted* QD (PQD) of order $n \in \mathbb{Z}_+$ if $\exists h \in \mathrm{Rat}_0(\Omega)$ (resp. $\mathrm{Rat}(\Omega)$) s.t

$$\int_\Omega f d\mu = \frac{1}{2\pi i} \oint_{\partial\Omega} f(w)h(w)dw, \qquad (d\mu(w) = n^2|w|^{2(n-1)}dA(w))$$

$\forall f \in \mathcal{A}(\Omega)$ (resp. $A_0(\Omega)$). Denoted by $\Omega \in \mathrm{QD}_n(h)$.

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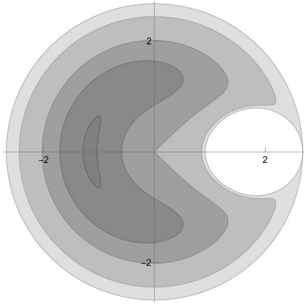
When $n = 1$, we recover classical QDs. On the other hand, when $n > 1$ then the metric is singular at 0. This results in interesting behaviour. For example when $\Omega \in \mathrm{QD}_2\left(\frac{10}{w-2}\right)$, a corner appears when $\partial\Omega$ intersects the origin.



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If $n \in \mathbb{Z}_+$ and $\Omega \in \mathrm{QD}_n(h)$ is bounded and s.c, then

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Similarly, if $\Omega \in \mathrm{QD}_n(h)$ is unbounded, then

$$\varphi^n(z) = W_n(z) - W_n(0) + \Phi_\varphi^{-1}\left(\frac{h + G(w)}{R'}\right)^{\#}(z)$$

for some $G \in \mathcal{A}_0(\Omega)$, where $W_n := \Phi_\varphi^{-1}(z^n)$ is the nth "inverse Faber polynomial".

If $0 \notin \Omega \in \mathrm{QD}_n\left(\frac{c}{w-w_0}\right)$ is bounded and s.c. with, $c > 0$, $w_0 \in \mathbb{C} \setminus \{0\}$ with Riemann map φ (wlog taking $\varphi(0) = w_0$, $\varphi'(0) > 0$), then

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So $\alpha = \sqrt{c}$. $\longrightarrow$ $\Omega = \left(\mathbb{D}_{\sqrt{c}}(w_0^n)\right)^{\frac{1}{n}}$ (taking the principal branch of $(\cdot)^{\frac{1}{n}}$).[17]

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Preview: Formulae for Higher Order One Point PQDs

Lemma 5.2

Let $n \in \mathbb{Z}_+$, $\Omega \ni w_0$ be a bounded, simply connected domain admitting a Riemann map $\varphi : \mathbb{D} \to \Omega$, $\varphi(0) = w_0$, and $X_k(w_0)$ the space of rational functions $= 0$ at ∞ with a unique pole of order $k \in \mathbb{Z}_+$ at w_0. Then,

1. *If $0 \notin \Omega$, then there exists $h \in X_k(w_0)$ such that $\Omega \in QD_n(h)$ if and only if there exist $z_1, \ldots, z_k \in \mathbb{D} \setminus \{0\}$ such that*
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Let Ω be an unbounded, simply connected domain with conformal radius a, admitting a Riemann map $\varphi : \mathbb{D}^- \to \Omega$. Then for each $n \in \mathbb{Z}_+$ and $k \in \mathbb{Z}_{\geq 0}$,

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That's all!

Thank you!